

Here,  $\zeta$  denotes the Riemann zeta function,  $\gamma$  is the Euler–Mascheroni constant, given by  $\gamma = \lim_{n \rightarrow \infty} (\sum_{k=1}^n 1/k - \log n)$ , and  $\gamma_1$  is the first Stieltjes constant, given by  $\gamma_1 = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{\log k}{k} - \frac{1}{2}(\log n)^2 \right)$ .

*Solution by Eugene A. Herman, Grinnell, IA.* By Euler's result (see G. T. Williams, A new method of evaluating  $\zeta(2n)$ , this MONTHLY **60** (1953) 19–25),

$$\sum_{m=1}^{k-2} \zeta(k-m)\zeta(m+1) = k\zeta(k+1) - 2 \sum_{j=2}^{\infty} \frac{H_{j-1}}{j^k},$$

where  $H_n$  is the  $n$ th harmonic number, given by  $H_n = \sum_{i=1}^n 1/i$ . By using  $\sum_{k=0}^{\infty} x^k = 1/(1-x)$  and  $\sum_{k=1}^{\infty} x^k/k = -\log(1-x)$  for  $|x| < 1$ , we obtain

$$\begin{aligned} & \sum_{k=3}^{\infty} \frac{1}{k} \left( \sum_{m=1}^{k-2} \zeta(k-m)\zeta(m+1) - k \right) \\ &= \sum_{k=3}^{\infty} \left( \zeta(k+1) - 1 - \frac{2}{k} \sum_{j=2}^{\infty} \frac{H_{j-1}}{j^k} \right) = \sum_{j=2}^{\infty} \sum_{k=3}^{\infty} \left( \frac{1}{j^{k+1}} - \frac{2H_{j-1}}{kj^k} \right) \\ &= \sum_{j=2}^{\infty} \left( \frac{1}{j^3(j-1)} + 2H_{j-1} \left( \log \left( 1 - \frac{1}{j} \right) + \frac{1}{j} + \frac{1}{2j^2} \right) \right). \end{aligned} \quad (1)$$

We evaluate each of the sums in (1):

$$\sum_{j=2}^{\infty} \frac{1}{j^3(j-1)} = \sum_{j=2}^{\infty} \left( \frac{1}{j-1} - \frac{1}{j} - \frac{1}{j^2} - \frac{1}{j^3} \right) = 3 - \frac{\pi^2}{6} - \zeta(3). \quad (2)$$

The partial sums of the next term telescope:

$$\begin{aligned} \sum_{j=2}^n H_{j-1} (\log(j-1) - \log j) &= \sum_{j=2}^{n-1} H_j \log j - \sum_{j=2}^n H_{j-1} \log j \\ &= \sum_{j=2}^n \frac{\log j}{j} - H_n \log n. \end{aligned} \quad (3)$$

By Euler's identity,  $\sum_{j=1}^{\infty} H_j/j^2 = 2\zeta(3)$ , so

$$\sum_{j=2}^{\infty} \frac{H_{j-1}}{j^2} = \sum_{j=1}^{\infty} \left( \frac{H_j}{j^2} - \frac{1}{j^3} \right) = \zeta(3). \quad (4)$$

We rearrange the terms in the partial sum of the remaining term:

$$\begin{aligned} \sum_{j=2}^n \frac{H_{j-1}}{j} &= \sum_{j=2}^n \frac{1}{j} \sum_{i=1}^{j-1} \frac{1}{i} = \sum_{i=1}^{n-1} \frac{1}{i} \sum_{j=i+1}^n \frac{1}{j} = \sum_{i=1}^{n-1} \frac{1}{i} (H_n - H_i) \\ &= H_n H_{n-1} - \sum_{i=1}^n \frac{H_i}{i} + \frac{H_n}{n} = H_n H_{n-1} - \sum_{i=1}^n \frac{H_{i-1}}{i} - \sum_{i=1}^n \frac{1}{i^2} + \frac{H_n}{n}. \end{aligned}$$

Bringing the second term on the right to the left side yields

$$2 \sum_{j=2}^n \frac{H_{j-1}}{j} = H_n H_{n-1} - \sum_{i=1}^n \frac{1}{i^2} + \frac{H_n}{n}. \quad (5)$$

Combining (1) through (5) and using  $\lim_{n \rightarrow \infty} H_n/n = 0$  yields

$$\begin{aligned} \sum_{k=3}^{\infty} \frac{1}{k} \left( \sum_{m=1}^{k-2} \zeta(k-m)\zeta(m+1) - k \right) &= 3 - \frac{\pi^2}{6} - \zeta(3) \\ &+ \lim_{n \rightarrow \infty} \left( 2 \sum_{j=2}^n \frac{\log j}{j} - 2H_n \log n + H_n H_{n-1} \right) - \frac{\pi^2}{6} + \zeta(3) \\ &= 3 - \frac{\pi^2}{3} + \lim_{n \rightarrow \infty} \left( 2 \sum_{j=2}^n \frac{\log j}{j} - \log^2 n - (H_n - \log n) (\log n - \log(n-1)) \right. \\ &\quad \left. + (H_n - \log n) (H_{n-1} - \log(n-1)) - \frac{\log n}{n} \right) \\ &= 3 - \frac{\pi^2}{3} + 2\gamma_1 + \gamma^2. \end{aligned}$$

*Editorial comment.* Mark W. Coffey proved alternating sum analogues of this result in *Sums of alternating products of Riemann zeta values and solution of a Monthly problem*, arXiv:1106.5147v2.

Also solved by M. W. Coffey, O. Kouba (Syria), M. R. Murty & A. Zaytseva (Canada), R. Stong, Ellington Management Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.