

Composite solution by Roberto Tauraso, Università di Roma “Tor Vergata”, Roma, Italy; Rituraj Nandan, St. Peters, MO; and the proposer. By d’Ocagne’s identity, (see <http://mathworld.wolfram.com/dOcagnesIdentity.html>), $F_N F_{M+1} - F_{N+1} F_M = (-1)^M F_{N-M}$. With $M = kn + 2k - 1$ and $N = k(n + 1) + 2k - 1$, we have

$$\begin{aligned} \frac{(-1)^{kn}}{F_{kn+2k-1} F_{kn+3k-1}} &= \frac{-F_{k(n+1)+2k-1} F_{kn+2k} + F_{k(n+1)+2k} F_{kn+2k-1}}{F_k F_{kn+2k-1} F_{k(n+1)+2k-1}} \\ &= \frac{1}{F_k} \left(\frac{F_{k(n+1)+2k}}{F_{k(n+1)+2k-1}} - \frac{F_{kn+2k}}{F_{kn+2k-1}} \right). \end{aligned}$$

Hence we have the telescoping sum

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^{kn}}{F_{kn+2k-1} F_{kn+3k-1}} &= \frac{1}{F_k} \sum_{n=0}^{\infty} \left(\frac{F_{k(n+1)+2k}}{F_{k(n+1)+2k-1}} - \frac{F_{kn+2k}}{F_{kn+2k-1}} \right) \\ &= \frac{1}{F_k} \left(\phi - \frac{F_{2k}}{F_{2k-1}} \right), \end{aligned}$$

where $\phi = \frac{1+\sqrt{5}}{2}$. Thus

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{a_k F_k F_{2k-1}}{2k-1} \sum_{n=0}^{\infty} \frac{(-1)^{kn}}{F_{kn+2k-1} F_{kn+3k-1}} &= \sum_{k=1}^{\infty} \frac{a_k (\phi F_{2k-1} - F_{2k})}{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{a_k}{2k-1} \left(\frac{1}{\phi} \right)^{2k-1}, \end{aligned}$$

since $F_{n+1} - \phi F_n = -\phi^{-n}$. The sequence $1, 2, 1, -1, -2, -1, \dots$ is the sum of two sequences whose cycles are $1, -1, 1, -1, 1, -1$ and $0, 3, 0, 0, -3, 0$. Continuing the computation,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{a_k}{2k-1} \left(\frac{1}{\phi} \right)^{2k-1} &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \left(\frac{1}{\phi} \right)^{2k-1} + \sum_{k=1}^{\infty} \frac{3(-1)^{k+1}}{6k-3} \left(\frac{1}{\phi} \right)^{6k-3} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \left(\frac{1}{\phi} \right)^{2k-1} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \left(\frac{1}{\phi^3} \right)^{2k-1} \\ &= \arctan \left(\frac{1}{\phi} \right) + \arctan \left(\frac{1}{\phi^3} \right). \end{aligned}$$

Using the arctangent addition formula and $\phi^2 = \phi + 1$, this becomes

$$\begin{aligned} \arctan \left(\frac{1}{\phi} \right) + \arctan \left(\frac{1}{\phi^3} \right) &= \arctan \left(\frac{\frac{1}{\phi} + \frac{1}{\phi^3}}{1 - \frac{1}{\phi} \cdot \frac{1}{\phi^3}} \right) = \arctan \left(\frac{\phi^3 + \phi}{\phi^4 - 1} \right) \\ &= \arctan \left(\frac{\phi^3 + \phi}{\phi^3 + \phi^2 - 1} \right) = \arctan \left(\frac{\phi^3 + \phi}{\phi^3 + \phi} \right) = \arctan(1) = \frac{\pi}{4}. \end{aligned}$$

Editorial comment. Omran Kouba proved the general identity

$$\arctan \left(\frac{2x \cos \theta}{1 - x^2} \right) = \sum_{k=1}^{\infty} \frac{2(-1)^{k-1} \cos((2k-1)\theta)}{2k-1} x^{2k-1}$$

for $\theta \in \mathbb{R}$ and $|x| < 1$, from which the second part of the proof above follows with $\theta = 2\pi/3$ and $x = 1/\phi$.

Also solved by R. Chapman (U. K.), O. Kouba (Syria), K. D. Lathrop, M. A. Prasad (India), R. Stong, S. Y. Xiao (Canada), and GCHQ Problem Solving Group (U. K.).